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Linear Algebra and Matrix Algebra Module

Linear equations are very common in Engineering and science problems. This tutorial will teach you how to use *Mathematica* to solve such equations. Given three linear equations,

4x + 3y + (-1) z = 9

5x + (-9) y+2 z = 3

x + y + z = 1

the unknowns *x*, *y*, and *z* can be solved directly with `Solve`. For example,

```
Clear[x, y, z]
Solve[{4 x + 3 y + (-1) z == 9, 5 x + (-9) y + 2 z == 3, x + y + z == 1},
  {x, y, z}]
```

$$\left\{\left\{x \to \frac{114}{67},\ y \to \frac{25}{67},\ z \to -\frac{72}{67}\right\}\right\}$$

However a more efficient approach is to take advantage of matrix mathematics. The unknowns *x*, *y*, and *z* in the previous equations can be written as,

$$\begin{pmatrix} 4 & 3 & -1 \\ 5 & -9 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix}$$

The three unknowns can be calculated directly with `LinearSolve`. `LinearSolve` works on both numerical and symbolic matrices, as well as `SparseArray` objects. For example,

```
M1 = {{4, 3, -1}, {5, -9, 2}, {1, 1, 1}};
M2 = {{9}, {3}, {1}};
xyzAnswer = LinearSolve[M1, M2];
Print["{x,y,z} = ", xyzAnswer]
```

$$\{x,y,z\} = \left\{\left\{\frac{114}{67}\right\}, \left\{\frac{25}{67}\right\}, \left\{-\frac{72}{67}\right\}\right\}$$

in which, M1 is the coefficient matrix and M2 is a column vector representing the right-hand-side of the equations. Note that the matrix can also be solved by multiplying the inversion of M1 and M2, however such an approach is far slower and sometimes inaccurate when the determinant of M1 is small.

We can check the result from `LinearSolve` by using the Dot product between the two matrices, M1 and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to obtain M2:

```
M1.xyzAnswer
```

```
{{9}, {3}, {1}}
```

It agrees with the value in M2. It is important to remember that `LinearSolve` will return one of the possible solutions for underdetermined linear equations however `Solve` will return a general solution.

Again, we can apply this approach to solve problem 4-14 where the systems of equations is given by:

$\frac{Cp}{T}$m + vαn = -s-pvα

vαm + vβn = -pvβ

```
M1 = {{Cp / T, v α}, {v α, v β}};
M2 = {{-s - p v α}, {-p v β}};
x = LinearSolve[M1, M2]
```

$$\left\{\left\{\frac{s\, T\, \beta}{T\, v\, \alpha^2 - Cp\, \beta}\right\}, \left\{\frac{-s\, T\, \alpha - p\, T\, v\, \alpha^2 + Cp\, p\, \beta}{T\, v\, \alpha^2 - Cp\, \beta}\right\}\right\}$$

Assuming that Cp = Cv + $T\, v\, \alpha^2 / \beta$, the solution simplifies as:

```
x /. {Cp → Cv + T v α^2 / β} // Simplify (*replace Cp with Cv+Tvα^2/β*)
```

$$\left\{\left\{-\frac{s\, T}{Cv}\right\}, \left\{-p + \frac{s\, T\, \alpha}{Cv\, \beta}\right\}\right\}$$

■ Inverse Approach

As mentioned previously, the Inverse Approach is far slower and sometimes inaccurate when the determinant of M1 is small. However, since this is the approach typically used in mathematics and engineering courses, it is useful to apply it. We will apply it to several examples.

Given two linear equations:

$$a\,m + b\,n = e \quad \& \quad c\,m + d\,n = f$$

and defining the unknowns, m and n,matrices can be defined:

$$M1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} m \\ n \end{pmatrix} \text{ and } M2 = \begin{pmatrix} e \\ f \end{pmatrix}$$

The unknowns m & n can then be solved for by using the following matrix methodology:

$$M1 \cdot X = M2$$

Finding the inverse matrix of M1, which is $M1^{-1}$, then we can solve for X :

$$X = M1^{-1} \cdot M2$$

In this example, we use a 3x3 matrices. That is, 3 equations and 3 unknowns.

```
Clear[a, b, c, d, M1, M2, M3, IM1, x1, x2, x1x2, M2, m, n]
M1 = {{4, 3, -1}, {5, -9, 2}, {1, 1, 1}}; IM1 = Inverse[M1]; b = {{x}, {y}, {z}}; M2 = {{9, 0, 0}, {3, 0, 0}, {1, 0, 0}};
Print["Inverse of M1, M1^-1 = ", MatrixForm[IM1]]
```

$$\text{Inverse of M1, } M1^{-1} = \begin{pmatrix} \frac{11}{67} & \frac{4}{67} & \frac{3}{67} \\ \frac{3}{67} & -\frac{5}{67} & \frac{13}{67} \\ -\frac{14}{67} & \frac{1}{67} & \frac{51}{67} \end{pmatrix}$$

Solving for X using :

$$X = M1^{-1} \cdot M2$$

We have :

```
Print["b = M1^-1.M2 = ", MatrixForm[IM1.M2], " =  ",
 MatrixForm[FullSimplify[IM1.M2]]]
```

$$b = M1^{-1}.M2 = \begin{pmatrix} \frac{114}{67} & 0 & 0 \\ \frac{25}{67} & 0 & 0 \\ -\frac{72}{67} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{114}{67} & 0 & 0 \\ \frac{25}{67} & 0 & 0 \\ -\frac{72}{67} & 0 & 0 \end{pmatrix}$$

We can apply this approach to sovle problem Problem 4 – 14 in DeHoff where 2 x2 matrices are used.

```
Clear[y, yi, m, n, z, Cp, Cv, mn, α, β, v, T, p]
y = ( Cp/T  v α
      v α   v β ); yi = Inverse[y]; x = ( m
                                          n ); z = ( -s - p v α
                                                     -p v β );
Print["M1 = ", MatrixForm[y], ";    X = ", MatrixForm[x], "; M2 = ",
 MatrixForm[z]]
Print["M1^-1= ", MatrixForm[yi], " = ", MatrixForm[FullSimplify[yi]]]
mn = yi.z; Cp = Cv + T v α^2/β;
Print["X=M1^-1·M2 = ", MatrixForm[mn]]
```

$$M1 = \begin{pmatrix} \frac{Cp}{T} & v\,\alpha \\ v\,\alpha & v\,\beta \end{pmatrix}; \quad X = \begin{pmatrix} m \\ n \end{pmatrix}; \; M2 = \begin{pmatrix} -s - p\,v\,\alpha \\ -p\,v\,\beta \end{pmatrix}$$

$$M1^{-1} = \begin{pmatrix} \frac{v\,\beta}{-v^2\,\alpha^2 + \frac{Cp\,v\,\beta}{T}} & -\frac{v\,\alpha}{-v^2\,\alpha^2 + \frac{Cp\,v\,\beta}{T}} \\ -\frac{v\,\alpha}{-v^2\,\alpha^2 + \frac{Cp\,v\,\beta}{T}} & \frac{Cp}{T\left(-v^2\,\alpha^2 + \frac{Cp\,v\,\beta}{T}\right)} \end{pmatrix} = \begin{pmatrix} \frac{T\,\beta}{-T\,v\,\alpha^2 + Cp\,\beta} & \frac{T\,\alpha}{T\,v\,\alpha^2 - Cp\,\beta} \\ \frac{T\,\alpha}{T\,v\,\alpha^2 - Cp\,\beta} & \frac{Cp}{v\,(-T\,v\,\alpha^2 + Cp\,\beta)} \end{pmatrix}$$

$$X = M1^{-1} \cdot M2 = \begin{pmatrix} \frac{p\,v^2\,\alpha\,\beta}{-v^2\,\alpha^2 + \frac{v\left(Cv + \frac{T\,v\,\alpha^2}{\beta}\right)\beta}{T}} + \frac{v\,(-s - p\,v\,\alpha)\,\beta}{-v^2\,\alpha^2 + \frac{v\left(Cv + \frac{T\,v\,\alpha^2}{\beta}\right)\beta}{T}} \\ -\frac{v\,\alpha\,(-s - p\,v\,\alpha)}{-v^2\,\alpha^2 + \frac{v\left(Cv + \frac{T\,v\,\alpha^2}{\beta}\right)\beta}{T}} - \frac{p\,v\left(Cv + \frac{T\,v\,\alpha^2}{\beta}\right)\beta}{T\left(-v^2\,\alpha^2 + \frac{v\left(Cv + \frac{T\,v\,\alpha^2}{\beta}\right)\beta}{T}\right)} \end{pmatrix}$$

Simplifying $X = M1^{-1} \cdot M2$, we have :

```
Print["X=M1^-1·M2 = ", MatrixForm[FullSimplify[mn]]]
```

$$X = M1^{-1} \cdot M2 = \begin{pmatrix} -\frac{s\,T}{Cv} \\ -p + \frac{s\,T\,\alpha}{Cv\,\beta} \end{pmatrix}$$